7.4 Lagrange Multipliers

Constrained optimization

Given a function $f(x,y)$ we want to min/max it under some constraint $g(x,y)=0$

Some examples: { • minimize surface area
• Constrained volume }

{ • Maximum height of mountain
• Constrained to a hiking trail }

Method: ① Set up a new function

$$F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

② Find derivatives

F_x, F_y, F_λ

③ Set $=0$ and solve for x, y

(If you get more than 1 answer, plug in and choose biggest/smallest.

ex/ Minimize

$$f(x,y) = x^2 + 3y^2 + 10$$

with constraint $\quad 8 = x + y$

$$\left(\text{so} \quad g(x,y) = 8 - x - y = 0 \right)$$

$$F(x, y, \lambda) = x^2 + 3y^2 + 10 + \lambda(8 - x - y)$$

$$F_x = 2x - \lambda$$

$$F_y = 6y - \lambda$$

$$F_\lambda = 8 - x - y$$

First 2 eq.

$$\begin{aligned} 2x - \lambda &= 0 \\ 6y - \lambda &= 0 \end{aligned} \quad \Rightarrow \quad \left.\begin{aligned} 2x &= \lambda \\ 6y &= \lambda \end{aligned}\right\} \text{set} = \text{to.}$$

$$2x = 6y$$

$$\Rightarrow x = 3y$$

plug into F_λ

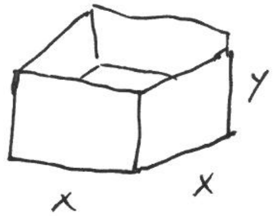
$$8 - 3y - y = 0$$

$$8 = 4y \quad \Rightarrow \quad \boxed{y = 2}$$

so $x = 3(2)$

$$\cancel{8 - x - 2} = 0 \quad \Rightarrow \quad \boxed{x = 6}$$

ex/ Maximize the volume of an open (no top) box with square base using 300 square inches of material



Maximize: $f(x,y) = x^2 y$

subject to: $4xy + x^2 - 300 = 0$

$$F(x,y,\lambda) = x^2 y + \lambda(4xy + x^2 - 300)$$

$$\begin{cases} F_x = 2xy + 4\lambda y + 2\lambda x = 0 \\ F_y = x^2 + 4\lambda x = 0 \end{cases}$$

$$F_\lambda = 4xy + x^2 - 300 = 0$$

$$\frac{-2xy}{4y+2x} = \lambda \quad \text{and} \quad \frac{-x^2}{4x} = \lambda$$

so $\quad \lambda = \dfrac{-xy}{2y+x}, \quad \lambda = \dfrac{-x}{4}$

$$\frac{xy}{2y+x} = \frac{x}{4}$$

$$4xy = 2xy + x^2$$

$$4y = 2y + x$$

$$\boxed{x = 2y}$$

$$0 = 4(2y)y + (2y)^2 - 300$$

$$300 = 12y^2$$

$$25 = y^2 \qquad \boxed{y = 5} \qquad \boxed{x = 10}$$

ex/ Find 2 positive #'s whose product is 25 and sum is small as possible.

Minimize $x+y$

Subject to $xy=25$ so $g(x,y)=xy-25$

$$F(x,y,\lambda)= x+y+\lambda(xy-25)$$

$$\left.\begin{aligned} F_x &= 1+\lambda y = 0 \\ F_y &= 1+\lambda x = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \lambda &= -\frac{1}{y} \\ \lambda &= -\frac{1}{x} \end{aligned}\right\} \quad -\frac{1}{y} = -\frac{1}{x} \Rightarrow x=y$$

$$F_\lambda = xy-25=0$$

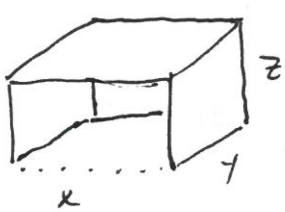
$$x^2-25=0$$

$\boxed{x=5}$ or $x=\cancel{-5}$ only positive

$\Rightarrow \boxed{y=5}$

Can do more than 2 variables

ex/ A bench shelter looks like a box with no floor or front made from 96 square feet of canvas. What dimensions maximize the volume.



maximize: xyz

subject to: $2yz+xz+xy-96=0$

$$F(x,y,z,\lambda) = xyz+\lambda(2yz+xz+xy-96)=0$$

$$F(x, y, z, \lambda) = xyz + \lambda(2yz + xz + xy - 96)$$

① $F_x = yz + \lambda z + \lambda y = 0$

② $F_y = xz + 2\lambda z + \lambda x = 0$

③ $F_z = xy + 2\lambda y + \lambda x = 0$

④ $F_\lambda = 2yz + xz + xy - 96 = 0$

From ①: $\dfrac{-yz}{z+y} = \lambda$

②: $\dfrac{-xz}{2z+x} = \lambda$

③: $\dfrac{-xy}{2y+x} = \lambda$

$\Rightarrow \dfrac{yz}{z+y} = \dfrac{xz}{2z+x}$

divide both by z

what if $z=0$?

$\dfrac{y}{z+y} = \dfrac{x}{2z+x}$

$\Rightarrow 2zy + \cancel{xy} = xz + \cancel{xy}$

$\Rightarrow \boxed{x = 2y}$

$\dfrac{xz}{2z+x} = \dfrac{xy}{2y+x}$

divide both by x
and cross-multiply

$\cancel{2yz} + zx = \cancel{2yz} + yx$

$\Rightarrow \boxed{y = z}$

plug $x = 2y$ and $z = y$
into constraint:

$2y(y) + 2y(y) + 2y(y) - 96 = 0$

$6y^2 = 96$

$y^2 = 16$

$\boxed{y = 4}$

$\boxed{z = 4}$

$\boxed{x = 8}$